Absolute (Global) Extreme Values

DEFINITION Absolute Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) **absolute maximum value** on D if and only if $f(x) \le f(c)$ for all x in D.

(b) **absolute minimum value** on D if and only if $f(x) \ge f(c)$ for all x in D.

EXAMPLE 1 Exploring Extreme Values

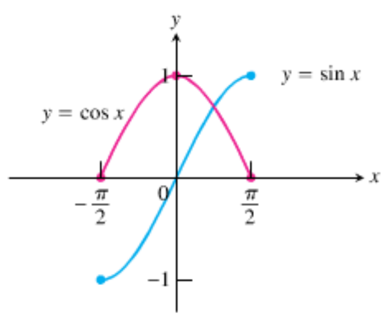
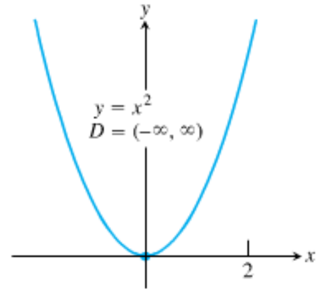
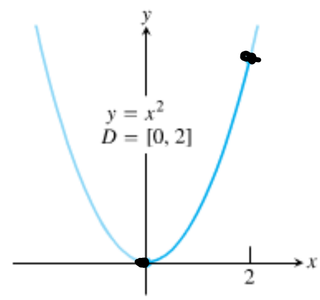


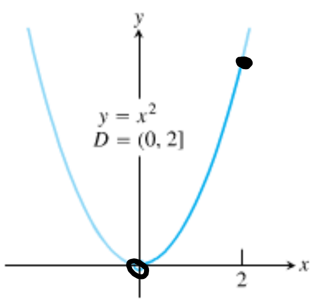
Figure 4.1 (Example 1)

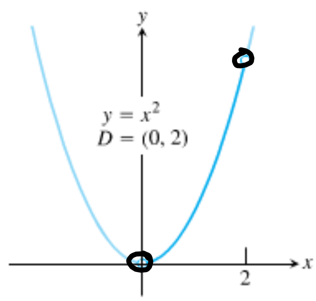
EXAMPLE 2 Exploring Absolute Extrema



	Function Rule	Domain D	Absolute Extrema on D
(a)	$y = x^2$	$(-\infty, \infty)$	min at $x=0$ of 0 no max
(b)	$y = x^2$	$[0, 2]$	max of 4 at $x=2$ min of 0 at $x=0$
(c)	$y = x^2$	$(0, 2]$	max of 4 at $x=2$ no min
(d)	$y = x^2$	$(0, 2)$	no max no min







Example 2 shows that a function may fail to have a maximum or minimum value. This cannot happen with a continuous function on a finite closed interval.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a maximum value and a minimum value on the interval. (Figure 4.3)

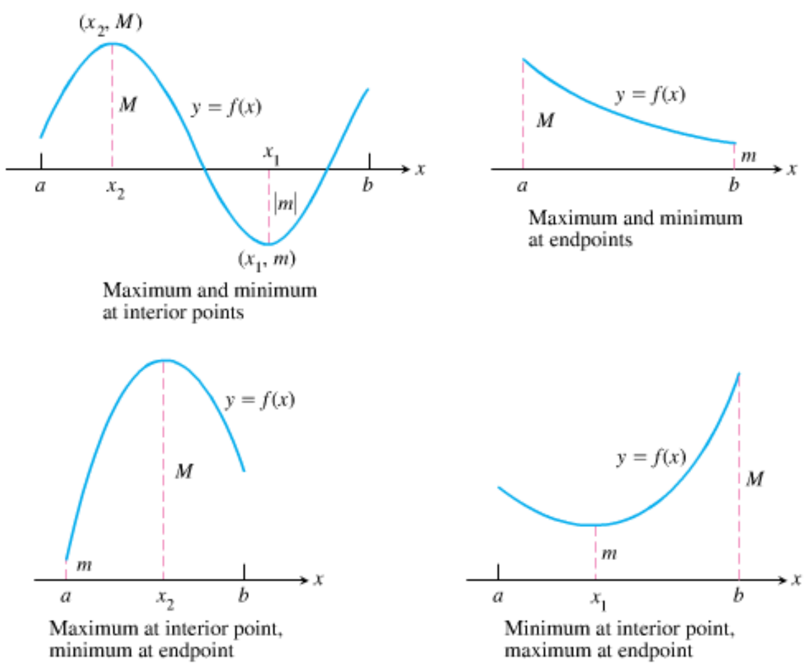


Figure 4.3 Some possibilities for a continuous function's maximum (M) and minimum (m) on a closed interval $[a, b]$.

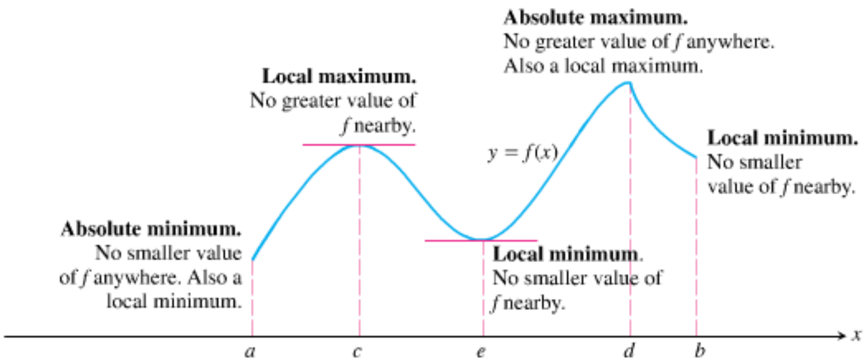


Figure 4.4 Classifying extreme values.

Local extrema are also called **relative extrema.**

DEFINITION Local Extreme Values

Let c be an interior point of the domain of the function f. Then $f(c)$ is a

(a) local maximum value at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

(b) local minimum value at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

A function f has a local maximum or local minimum *at an endpoint* c if the appropriate inequality holds for all x in some half-open domain interval containing c.

THEOREM 2 Local Extreme Values

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0.$$

An **absolute extremum** is also a local extremum.

DEFINITION Critical Point

A point in the interior of the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

DEFINITION Stationary Point

A point in the interior of the domain of a function f at which $f' = 0$ is called a ***stationary point*** of f.

EXAMPLE 3 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$$

$$= \frac{2}{3\sqrt[3]{x}}$$

undefined: at $x=0$

critical point $(0,0)$

value → 0

endpoints

$f(-2) = \sqrt[3]{4}$

$f(3) = \sqrt[3]{9}$

absolute max $= \sqrt[3]{9}$ at $x=3$

absolute min $= 0$ at $x=0$

EXAMPLE 4 Finding Extreme Values

Find the extreme values of $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$.

domain: $4 - x^2 > 0$ $(-2, 2)$

$x^2 < 4$

$x < 2$

$x > -2$

Critical points:

$f(x) = (4 - x^2)^{-1/2}$

$f'(x) = -\frac{1}{2}(4 - x^2)^{-3/2}(-2x)$

$= \dfrac{x}{(4 - x^2)^{3/2}}$

$f'(x) = 0$ when $x = 0$

value $= \frac{1}{2}$

$-$ $+$

-2 0 2

undefined: at $x = -2$ & $x = 2$ not in the domain!

~~local~~ min of $1/2$ at $x = 0$

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value.

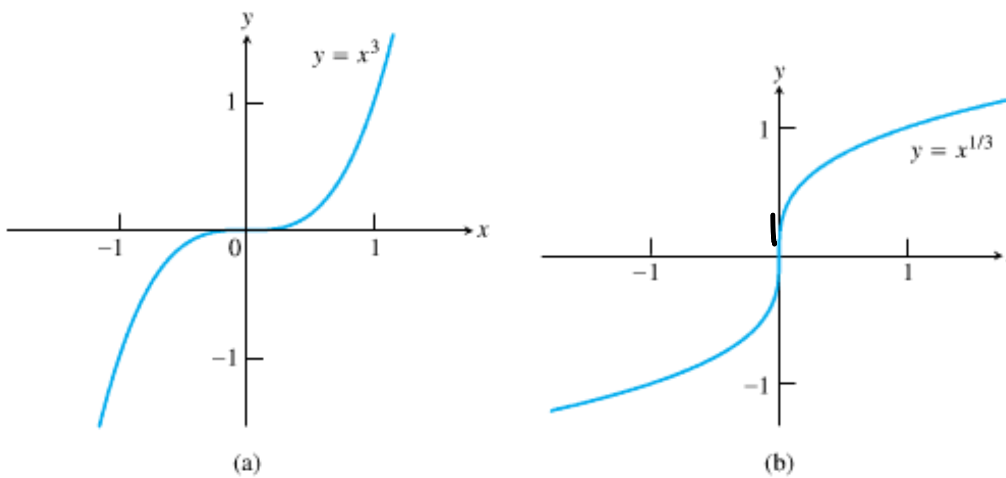


Figure 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

EXAMPLE 5 Finding Extreme Values

Find the extreme values of

$$f(x) = \begin{cases} 5 - 2x^2, & x \le 1 \\ x + 2, & x > 1. \end{cases}$$

EXAMPLE 6 Using Graphical Methods

Find the extreme values of $f(x) = \ln\left|\frac{x}{1+x^2}\right|$.

SOLUTION